

Physical Realizability of Multi-Level Quantum Systems*

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Abstract—This paper considers the physical realizability condition for multi-level quantum systems having polynomial Hamiltonian and multiplicative coupling with respect to several interacting boson fields. Specifically, it generalizes a recent result the authors developed for two-level quantum systems. For this purpose, the algebra of $SU(n)$ was incorporated. As a consequence, the obtained condition is given in terms of the structure constants of $SU(n)$.

I. INTRODUCTION

In an environment where the classical laws of physics apply, standard control techniques such as optimization or a Lyapunov procedures do not worry in general of the nature of the controller they synthesized. In other words, their implementation is always possible since the physics behind them still hold. However, if one desires to implement a controller that obeys the laws imposed by quantum mechanics (e.g., quantum coherent control [2], [7], [12]), then such a task is not so easily achieved unless an explicit characterization of those laws is given in terms of the control system vector fields. This is exactly the purpose for introducing the concept of *physical realizability*.

Conditions for physical realizability were first given specifically for linear systems satisfying the quantum harmonic oscillator canonical commutation relations [5], [8]. Recently, the formalism was extended for systems describing the dynamics of open two-level quantum systems interacting only with one quantum field in which the algebra of $SU(2)$ played a central role [3]. Compared to a linear quantum system, the systems being analyzed were bilinear, and the the commutation relations were dependent on the system variables. Thus, the main contribution of this paper, given in Section IV, is to provide a condition for physical realizability of multi-level quantum systems having polynomial Hamiltonian and multiplicative coupling, and whose system variables obey the commutation relations described by the algebra of $SU(n)$. As expected, the obtained condition is given in terms of the symmetric and antisymmetric structure constants of $SU(n)$. Another contribution is that the systems under consideration have been allowed to interact with multiple quantum fields in quadrature form.

The paper is organized as follows. Section II presents the basic preliminaries on open quantum systems. In Section III,

the necessary algebraic machinery to study open multi-level quantum systems is given. This is followed by Section IV, in which the definition of physical realizability is provided as well as a condition for a bilinear QSDE to be physically realizable. Finally, Section V gives the conclusions.

II. OPEN MULTI-LEVEL QUANTUM SYSTEMS

Open quantum systems are systems governed by the laws of quantum mechanics that interact with an external environment. A quantum mechanical system is described in terms of *observables* and *states*. Observables represent physical quantities that can be measured, as self-adjoint operators on a complex separable Hilbert space \mathfrak{H} , while states give the current status of the system, as elements of \mathfrak{H} , allowing the computation of expected values of observables. In [1], [11], the evolution of open quantum systems is given in terms of quantum stochastic differential equations. For this purpose, observables may be thought as quantum random variables that do not in general commute. A measure of the non commutativity between observables is usually given by the *commutator* between operators.

Definition 1: The commutator of two scalar operators x and y in \mathfrak{H} is the antisymmetric bilinear operation

$$[x, y] = xy - yx.$$

Also, if x is an n_1 -dimensional vector of operators in \mathfrak{H} and y is an n_2 -dimensional vector of operators in \mathfrak{H} , then

$$[x, y^T] \triangleq xy^T - (yx^T)^T,$$

which is an $n_1 \times n_2$ matrix of operators in \mathfrak{H} .

This commutator satisfies

$$[x, y^T]^T = -yx^T + (xy^T)^T = -[y, x^T]. \quad (1)$$

The adjoint of x is denoted by $x^\dagger = (x^\#)^T$ with

$$x^\# \triangleq \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$$

and $*$ denotes the operator adjoint. In the case of complex vectors and matrices, $*$ denotes the complex conjugate while † denotes the conjugate transpose. The non-commutativity of observables is a fundamental difference between quantum systems and classical systems in which the former must satisfy certain commutation relations originating from *Heisenberg uncertainty principle*.

The environment consists of a collection of oscillator systems each with annihilation field operator $w(t)$ and creation field operator $w^*(t)$ used for the annihilation and creation of quanta at point t , and commonly known as the *boson*

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quantum field (with parameter t). Here it is assumed that t is a real time parameter. The field operators $w(t)$ and $w^*(t)$ satisfy commutation relations as well. That is,

$$[w(t), w^*(t')] = \delta(t - t')$$

for all $t, t' \in \mathbb{R}$, where $\delta(t)$ denotes the Dirac delta. Its mathematical description is given in terms of a Hilbert space called a *Fock space*. When the boson quantum field is in the vacuum state, i.e., no physical particles are present, it then represents a natural quantum extension of white noise, and may be described using the quantum Itô calculus [1], [11]. This amounts to having three interacting signals (inputs) in the evolution of the system: the annihilation processes $W(t)$, the creation process $W^*(t)$, and the counting process $\Lambda_w(t)$.

In simple words, the evolution of an open quantum system is described putting together the evolutions of the system and the environment in an unitary fashion. That is, if ψ is an initial state then $\psi(t) = U(t)\psi$, where $U(t)$ is unitary for all t , and is the solution of

$$dU(t) = \left((S - I) d\Lambda_w(t) + L dW^*(t) - L^* S dW(t) - \frac{1}{2}(L^* L + \mathbf{i}\mathcal{H}) dt \right) U(t),$$

with initial condition $U(0) = I$, I denoting the identity operator and \mathbf{i} being the imaginary unit. Here, \mathcal{H} is a fixed self-adjoint operator representing the *Hamiltonian* of the system, and L and S are operators determining the *coupling* of the system to the field, with S unitary. The evolution of ψ is equivalent to the evolution of the observable X given by

$$X(t) = U^*(t)(X \otimes I)U(t),$$

whose evolution is referred as the *Heisenberg picture* while the one for ψ is known as the *Schrödinger picture*. This paper exclusively takes the point of view of the Heisenberg picture. The quantum stochastic calculus in [11] allows to express the Heisenberg picture evolution of an scalar operator X interacting with a boson field as

$$dX = (S^* X S - X) d\Lambda_w + \mathcal{L}(X) dt + S^* [X, L] dW^* + [L^*, X] S dW,$$

where $\mathcal{L}(X)$ is the Lindblad operator defined as

$$\mathcal{L}(X) = -\mathbf{i}[X, \mathcal{H}] + \frac{1}{2}(L^* [X, L] + [L^*, X] L).$$

The output field is given by

$$Y(t) = U(t)^* W(t) U(t),$$

which amounts to

$$dY = L dt + S dW.$$

In summary, the dynamics of an open quantum system is uniquely determined by the triple (S, L, \mathcal{H}) . Hereafter, the operator S is assumed to be the identity operator ($S = I$). If on the other hand one consider n_w interacting boson fields then the evolution equation is written as

$$dX = \mathcal{L}(X) dt + dW^\dagger [X, L] + [L^\dagger, X] dW,$$

where $[X, dW] = [X, dW^\dagger]^T = 0$, $L = (L_1, \dots, L_s)^T$,

$$\mathcal{L}(X) \triangleq -\mathbf{i}[X, H] + \frac{1}{2}(L^\dagger [X, L] + [L^\dagger, X] L),$$

$$dW = \begin{pmatrix} dW_1 \\ \vdots \\ dW_{n_w} \end{pmatrix} \text{ and } dW^\dagger = (dW_1^*, \dots, dW_{n_w}^*).$$

Consider the vector of operators $x = (x_1, \dots, x_s)^T$. By stacking (column-wise) the scalar evolutions for each x_i , it follows that the Heisenberg evolution equation is

$$\begin{aligned} \begin{pmatrix} dx_1 \\ \vdots \\ dx_s \end{pmatrix} &= \begin{pmatrix} \mathcal{L}(x_1) \\ \vdots \\ \mathcal{L}(x_s) \end{pmatrix} dt + \begin{pmatrix} [x_1, L^T] \\ \vdots \\ [x_s, L^T] \end{pmatrix} dW^\# \\ &\quad + \begin{pmatrix} [L^\dagger, x_1] \\ \vdots \\ [L^\dagger, x_s] \end{pmatrix} dW \\ &= \begin{pmatrix} \mathcal{L}(x_1) \\ \vdots \\ \mathcal{L}(x_s) \end{pmatrix} dt + \begin{pmatrix} [x_1, L^T] \\ \vdots \\ [x_s, L^T] \end{pmatrix} dW^\# \\ &\quad - \begin{pmatrix} [x_1, L^\dagger] \\ \vdots \\ [x_s, L^\dagger] \end{pmatrix} dW \\ &= \mathcal{L}(x) dt + [x, L^T] dW^\# - [x, L^\dagger] dW, \end{aligned} \quad (2)$$

where

$$\begin{aligned} \mathcal{L}(x) &= -\mathbf{i}[x, H] \\ &\quad + \frac{1}{2} \left(\begin{pmatrix} L^\dagger [x_1, L] \\ \vdots \\ L^\dagger [x_s, L] \end{pmatrix} + \begin{pmatrix} [L^\dagger, x_1] L \\ \vdots \\ [L^\dagger, x_s] L \end{pmatrix} \right) \\ &= -\mathbf{i}[x, H] - \frac{1}{2} \left((L^\dagger ([L, x_1], \dots, [L, x_s]))^T \right) \\ &\quad - \frac{1}{2} [x, L^\dagger] L \\ &= -\mathbf{i}[x, H] + \frac{1}{2} \left(- (L^\dagger [L, x^T])^T - [x, L^\dagger] L \right) \\ &= -\mathbf{i}[x, H] + \frac{1}{2} \left((L^\dagger [x, L^T]^T)^T + [L^\dagger, x^T]^T L \right). \end{aligned} \quad (3)$$

It is customary to express QSDEs in terms of its interaction with quadrature fields. The quadrature fields are given by the transformation

$$\begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \end{pmatrix} = \begin{pmatrix} I_{n_w} & I_{n_w} \\ -\mathbf{i}I_{n_w} & \mathbf{i}I_{n_w} \end{pmatrix} \begin{pmatrix} W \\ W^\# \end{pmatrix}, \quad (4)$$

where the operators \bar{W}_1 and \bar{W}_2 are now self-adjoint, and I_{n_w} denotes the identity matrix of dimension n_w . In [4], the Itô table for W and W^\dagger is

$$\begin{pmatrix} dW \\ dW^\# \end{pmatrix} \begin{pmatrix} dW & dW^\# \end{pmatrix} = \begin{pmatrix} 0 & I_{n_w} \\ 0 & 0 \end{pmatrix} dt,$$

which in terms of the quadrature fields is

$$\begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix} \begin{pmatrix} d\bar{W}_1 & d\bar{W}_2 \end{pmatrix} = \begin{pmatrix} I_{n_w} & \mathbf{i}I_{n_w} \\ -\mathbf{i}I_{n_w} & I_{n_w} \end{pmatrix} dt.$$

Thus,

$$\begin{aligned} dx &= \mathcal{L}(x) dt \\ &+ \frac{1}{2} ([x, L^T], -[x, L^\dagger]) \begin{pmatrix} I_{n_w} & -\mathbf{i}I_{n_w} \\ I_{n_w} & \mathbf{i}I_{n_w} \end{pmatrix} \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix} \\ &= \mathcal{L}(x) dt + \frac{1}{2} ([x, L^T] - [x, L^\dagger]) d\bar{W}_1 \\ &- \frac{\mathbf{i}}{2} ([x, L^T] + [x, L^\dagger]) d\bar{W}_2 \end{aligned}$$

The quadrature form of the output fields is obtained from the quadrature transformation

$$\begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{pmatrix} = \begin{pmatrix} I_{n_w} & I_{n_w} \\ -\mathbf{i}I_{n_w} & \mathbf{i}I_{n_w} \end{pmatrix} \begin{pmatrix} Y \\ Y^\dagger \end{pmatrix},$$

which gives

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} L + L^\# \\ \mathbf{i}(L^\# - L) \end{pmatrix} dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}.$$

The main focus of this paper is on the dynamics of open multi-level quantum systems interacting with n_w Boson quantum fields. Such systems evolve with respect to the group $SU(n)$. The algebra of $SU(n)$ has been extensively studied since the 1950's to the point that it is an standard topic in quantum mechanics when studying multi-level systems [9], [10]. To particularize the framework presented in the previous paragraph for system (2) evolving on $SU(n)$, consider the Hilbert space $\mathfrak{H} = \mathbb{C}^n$ and let $|j\rangle$ with $j = 1, \dots, n$ be eigenvectors spanning \mathfrak{H} . A *projection operator* P_{kl} is defined as the outer product

$$P_{k,l} = |k\rangle \langle l|,$$

where $k, l = 1, \dots, n$. It is a well-known fact that any operator defined in \mathfrak{H} can be obtained in term of these n^2 projection operators. Specifically the generators of $SU(n)$ are constructed as follows

$$\begin{aligned} u_{jk} &= P_{j,k} + P_{k,j}, \\ v_{jk} &= \mathbf{i}(P_{j,k} - P_{k,j}), \\ w_l &= -\sqrt{\frac{2}{l(l+1)}} \left(\sum_{s=1}^k P_{s,s} - kP_{l+1,l+1} \right) \end{aligned}$$

for $1 \leq j < k \leq n$, $1 \leq l \leq n-1$. The $(n^2-n)/2$ symmetric matrices u_{jk} , the $(n^2-n)/2$ antisymmetric matrices v_{jk} and the $n-1$ mutually commutative matrices w_l together form the set $\{\lambda_1, \dots, \lambda_{n^2-1}\}$ of generators of $SU(n)$. Hereafter define $s = n^2 - 1$. Their commutation and anticommutation relations are

$$\begin{aligned} [\lambda_i, \lambda_j] &= 2\mathbf{i} \sum_{k=1}^s f_{ijk} \lambda_k, \\ \{\lambda_i, \lambda_j\} &= \frac{4}{n} \delta_{ij} + 2 \sum_{k=1}^s d_{ijk} \lambda_k. \end{aligned}$$

Thus, the product $\lambda_i \lambda_j$ can be easily computed as

$$\lambda_i \lambda_j = \frac{1}{2} ([\lambda_i, \lambda_j] + \{\lambda_i, \lambda_j\})$$

$$= \frac{2}{n} \delta_{ij} + \sum_{k=1}^s (\mathbf{i}f_{ijk} + d_{ijk}) \lambda_k. \quad (5)$$

where the real completely antisymmetric tensor f_{ijk} and the real completely symmetric tensor d_{ijk} are called the *structure constants* of $SU(n)$, and δ_{ij} is the Kronecker delta. The tensors f_{ijk} and d_{ijk} satisfy

$$f_{ilm} f_{mjk} + f_{jlm} f_{imk} + f_{klm} f_{ijm} = 0, \quad (6)$$

$$f_{ilm} d_{mjk} + f_{jlm} d_{imk} + f_{klm} d_{ijm} = 0, \quad (7)$$

and

$$\sum_{m,k=1}^s f_{imk} f_{jmk} = n \delta_{ij}. \quad (8)$$

The procedure of how to construct the generalized Gell-Mann matrices shows that only the dimension of the group $SU(n)$ is necessary to express all the components of the group algebra, i.e., once n is given then the generators and structure constants f and d are fixed. The vector of system variables for (2) is

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} \triangleq \begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_s \end{pmatrix},$$

where $\hat{\lambda}_1, \dots, \hat{\lambda}_s$ are spin operators. Given that these operators are self-adjoint, the vector of operators x satisfies $x = x^\#$. In particular, a self-adjoint operator $\hat{\lambda}$ in \mathfrak{H} is spanned by the generalized Gell-Mann matrices [10], i.e.,

$$\hat{\lambda} = \frac{1}{n} \alpha_0 + \frac{1}{2} \sum_{i=0}^s \alpha_i \lambda_i,$$

where $\alpha_0 = \text{Tr}(\hat{\lambda})$, $\alpha_i = \text{Tr}(\hat{\lambda} \lambda_i)$. Thus, α_0 and $(\alpha_1, \dots, \alpha_s)^T \in \mathbb{C}^3$ determine uniquely the operator $\hat{\lambda}$ with respect to a given basis in \mathbb{C}^n . The initial value of the system variables can be set to $x(0) = (\lambda_1, \dots, \lambda_s)$.

It is important to emphasize that any arbitrary polynomial of the components of x evolving in $SU(n)$ is described by a linear combination of the generators of $SU(n)$, which includes the identity [10]. This fact can be appreciated in relation (5) because equation (2) implies that such commutation relations are preserved by the evolution in time. Thus, any Hamiltonian and coupling operators of polynomial type are representable as linear functions of x . Therefore, assuming linearity captures a large class of Hamiltonian and coupling operators without much loss of generality, i.e., the assumed Hamiltonian is $\mathcal{H} = \alpha x$ with $\alpha \in \mathbb{R}^s$, and the multiplicative coupling operator is of the form $L = \Lambda x$ with $\Lambda \in \mathbb{C}^{n_w \times s}$. The reason why a coupling operator is called multiplicative is that they make the interacting fields to appear in (2) as multiplicative quantum noise.

In general, the evolution of x in quadrature form falls into a class of bilinear QSDEs expressed as

$$\begin{aligned} dx &= A_0 dt + A x dt \\ &+ (B_{11}x, \dots, B_{1n_w}x, B_{21}x, \dots, B_{2n_w}x) \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \quad (9) \end{aligned}$$

where $A_0 \in \mathbb{R}^s$ and $A, B_{1k} \triangleq \bar{B}_{1k} + \bar{B}_{2k}, B_{2k} \triangleq \mathbf{i}(\bar{B}_{2k} - \bar{B}_{1k}) \in \mathbb{R}^{s \times s}$, $k = 1, \dots, n_w$. The fact that all matrices in (9) are real is due to the quadrature transformation (4). The quadrature output fields are whose quadrature form is

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \quad (10)$$

where $C \in \mathbb{C}^{n_w \times s}$, $C_1 \triangleq C + C^\#$ and $C_2 \triangleq \mathbf{i}(C^\# - C)$. Note that the quadrature transformation makes C_1 and C_2 to be real matrices.

The objective of this paper is to determine conditions on the coefficients in (9) and (10) under which there exists the Hamiltonian and the coupling operator of the above form such that (9), (10) can be written as 2; see Definition 3 in Section IV.

III. NOTATION AND PRELIMINARY RESULTS

Define $F_i, D_i \in \mathbb{R}^{s \times s}$, $i \in \{1, \dots, s\}$, such that their (j, k) component is $(F_i)_{jk} = f_{ijk}$ and $(D_i)_{jk} = d_{ijk}$, respectively. In particular, the set $\{-\mathbf{i}F_1, \dots, -\mathbf{i}F_s\}$ is the adjoint representation of $SU(n)$. In [6], [9], identities (6) and (7) were employed to obtain the following useful relationships

$$[F_i, F_j] = - \sum_k f_{ijk} F_k \quad (11)$$

$$[F_i, D_j] = - \sum_k f_{ijk} D_k \quad (12)$$

$$F_i D_j + F_j D_i = \sum_k d_{ijk} F_k \quad (13)$$

$$D_i F_j + D_j F_i = \sum_k d_{ijk} F_k. \quad (14)$$

Definition 2: Let $\beta \in \mathbb{C}^s$. The linear mappings $\Theta^-, \Theta^+ : \mathbb{C}^s \rightarrow \mathbb{C}^{s \times s}$ are defined as

$$\Theta^-(\beta) = (F_1^T \beta, \dots, F_s^T \beta) = \begin{pmatrix} \beta^T F_1^T \\ \vdots \\ \beta^T F_s^T \end{pmatrix}, \quad (15)$$

$$\Theta^+(\beta) = (D_1^T \beta, \dots, D_s^T \beta) = \begin{pmatrix} \beta^T D_1^T \\ \vdots \\ \beta^T D_s^T \end{pmatrix}. \quad (16)$$

In order to simplify the notation, if β is an s -dimensional row then it will be understood hereafter that $\Theta^-(\beta) = \Theta^-(\beta^T)$ and $\Theta^+(\beta) = \Theta^+(\beta^T)$. In addition, these two matrix functions are used to express the commutation and anticommutation relations of the vector of operators x in a compact form. That is,

$$[x, x^T] = 2\mathbf{i}\Theta^-(x), \quad (17)$$

$$\{x, x^T\} = \frac{4}{n}I + 2\Theta^+(x). \quad (18)$$

It is also important to notice that the nature of the f and d -tensors make the matrices $\Theta^-(\beta)$ and $\Theta^+(\beta)$ be antisymmetric and symmetric, respectively. Consider now the *stacking operator* $\text{vec} : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{mn}$ whose action on a

matrix creates a column vector by stacking its columns below one another. With the help of vec , the matrices $\Theta^-(\beta)$ and $\Theta^+(\beta)$ can be reorganized so that

$$\text{vec}(\Theta^-(\beta)) = \begin{pmatrix} \Theta_1^-(\beta) \\ \vdots \\ \Theta_s^-(\beta) \end{pmatrix} = F\beta,$$

and

$$\text{vec}(\Theta^+(\beta)) = \begin{pmatrix} \Theta_1^+(\beta) \\ \vdots \\ \Theta_s^+(\beta) \end{pmatrix} = D\beta,$$

where $\beta \in \mathbb{C}^s$, $\Theta_i^-(\beta) = F_i^T \beta$, $F = (F_1, \dots, F_s)^T$, $\Theta_i^+(\beta) = D_i \beta$ and $D = (D_1, \dots, D_s)^T$. From (8), F satisfies

$$\begin{aligned} (F^T F)_{ij} &= \left(\sum_{k=1}^s F_k F_k^T \right)_{ij} \\ &= - \sum_{k,m=1}^s (F_k)_{im} (F_k)_{mj} \\ &= - \sum_{k,m=1}^s f_{kim} f_{kmj} \\ &= \sum_{k,m=1}^s f_{imk} f_{jmk} \\ &= n\delta_{ij}, \end{aligned}$$

which implies

$$F^T F = nI. \quad (19)$$

The properties of Θ^- and Θ^+ are summarized in the next lemma.

Lemma 1: Let $\beta, \gamma \in \mathbb{C}^s$. The mappings Θ^- and Θ^+ satisfy

- i. $\Theta^-(\beta)\gamma = -\Theta^-(\gamma)\beta$,
- ii. $\Theta^+(\beta)\gamma = \Theta^+(\gamma)\beta$,
- iii. $\Theta^-(\beta)\beta = 0$,
- iv. $\Theta^-(\Theta^-(\beta)\gamma) = [\Theta^-(\beta), \Theta^-(\gamma)]$,
- v. $\Theta^-(\Theta^+(\beta)\gamma) = \{\Theta^-(\beta), \Theta^+(\gamma)\}$,
- vi. $\Theta^+(\Theta^-(\beta)\gamma) = [\Theta^+(\beta), \Theta^-(\gamma)] = [\Theta^-(\beta), \Theta^+(\gamma)]$,

where the commutator of matrices is defined as usual, i.e., $[A, B] = AB - BA$ for $A, B \in \mathbb{R}^{s \times s}$.

Proof: Using (15), one can decompose the left-hand-side of (i) in terms of the matrices F_i as

$$\Theta^-(\beta)\gamma = \begin{pmatrix} \beta^T F_1^T \gamma \\ \vdots \\ \beta^T F_s^T \gamma \end{pmatrix}$$

Every component is the written as

$$\beta^T F_i^T \gamma = - \begin{pmatrix} \sum_{k,l=1}^s \beta_l f_{ilk} \gamma_k \\ \vdots \\ \sum_{k,l=1}^s \beta_l f_{ilk} \gamma_k \end{pmatrix} = \begin{pmatrix} \sum_{k,l=1}^s \gamma_k f_{ikl} \beta_l \\ \vdots \\ \sum_{k,l=1}^s \gamma_k f_{ikl} \beta_l \end{pmatrix},$$

which implies $\beta^T F_i^T \gamma = -\gamma^T F_i^T \beta$. Therefore $\Theta^-(\beta)\gamma = -\Theta^-(\gamma)\beta$. Using (16), a similar procedure is applied for identity (ii) in terms of the matrices D_i . The i -th component is given by

$$\beta^T D_i \gamma = \begin{pmatrix} \sum_{k,l=1}^s \beta_l d_{ilk} \gamma_k \\ \vdots \\ \sum_{k,l=1}^s \beta_l d_{ilk} \gamma_k \end{pmatrix} = \begin{pmatrix} \sum_{k,l=1}^s \gamma_k d_{ikl} \beta_l \\ \vdots \\ \sum_{k,l=1}^s \gamma_k d_{ikl} \beta_l \end{pmatrix},$$

which gives $\beta^T D_i \gamma = \gamma^T D_i \beta$. Thus, $\Theta^+(\beta)\gamma = \Theta^+(\gamma)\beta$. Identity (iii) is true since $f_{ijj} = 0$ for all i and j , and

$$\begin{aligned} \sum_{k,l=1}^s \beta_l f_{ilk} \beta_k &= \sum_{\substack{k,l=1 \\ k \neq l}}^s \beta_l f_{ilk} \beta_k \\ &= \sum_{k < l} \beta_l f_{ilk} \beta_k + \sum_{k > l} \beta_l f_{ilk} \beta_k \\ &= \sum_{k < l} \beta_l f_{ilk} \beta_k - \sum_{k < l} \beta_l f_{ilk} \beta_k \\ &= 0, \end{aligned}$$

where the negative sign in the last summand was obtained because of the antisymmetry of f_{ilk} . The left-hand-side of identity (iv) is decomposed as

$$\begin{aligned} &\Theta^-(\Theta^-(\beta)\gamma) \\ &= \left(F_1 \begin{pmatrix} \beta^T F_1 \gamma \\ \vdots \\ \beta^T F_s \gamma \end{pmatrix}, \dots, F_s \begin{pmatrix} \beta^T F_1 \gamma \\ \vdots \\ \beta^T F_s \gamma \end{pmatrix} \right) \\ &= \begin{pmatrix} \sum_{k=1}^s f_{11k} \beta^T F_k \gamma & \cdots & \sum_{k=1}^s f_{s1k} \beta^T F_k \gamma \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^s f_{1sk} \beta^T F_k \gamma & \cdots & \sum_{k=1}^s f_{ssk} \beta^T F_k \gamma \end{pmatrix}. \end{aligned}$$

By (11), the (i, j) component of this matrix is

$$\begin{aligned} (\Theta^-(\Theta^-(\beta)\gamma))_{ij} &= - \sum_{k=1}^s f_{ijk} \beta^T F_k \gamma \\ &= \beta^T \left(- \sum_{k=1}^s f_{ijk} F_k \right) \gamma \\ &= \beta^T [F_i, F_j] \gamma \\ &= \beta^T F_i F_j \gamma - \beta^T F_j F_i \gamma \\ &= \beta^T F_i^T F_j^T \gamma - \beta^T F_j^T F_i^T \gamma \end{aligned}$$

$$\begin{aligned} &= \beta^T F_i^T F_j^T \gamma - \gamma^T F_i^T F_j^T \beta \\ &= (\Theta^-(\beta)\Theta^-(\gamma) - \Theta^-(\gamma)\Theta^-(\beta))_{ij} \\ &= ([\Theta^-(\beta), \Theta^-(\gamma)])_{ij}. \end{aligned}$$

Similarly, decomposing the (i, j) component of the left-hand-side of (v) and using (12) gives

$$\begin{aligned} (\Theta^-(\Theta^+(\beta)\gamma))_{ij} &= \sum_{k=1}^s f_{ijk} \beta^T D_k \gamma \\ &= -\beta^T \left(- \sum_{k=1}^s f_{ijk} D_k \right) \gamma \\ &= -\beta^T [F_i, D_j] \gamma \\ &= -\beta^T F_i D_j \gamma + \beta^T D_j F_i \gamma \\ &= \beta^T F_i^T D_j \gamma - \beta^T D_j F_i^T \gamma \\ &= \beta^T F_i^T D_j \gamma + \gamma^T F_i^T D_j \beta \\ &= (\Theta^-(\beta)\Theta^+(\gamma) + \Theta^-(\gamma)\Theta^+(\beta))_{ij} \\ &= (\{\Theta^-(\beta), \Theta^+(\gamma)\})_{ij}. \end{aligned}$$

Again, decomposing the (i, j) component of the left-hand-side of (vi) and using (13) gives

$$\begin{aligned} (\Theta^+(\Theta^-(\beta)\gamma))_{ij} &= - \sum_{k=1}^s d_{ijk} \beta^T F_k \gamma \\ &= -\beta^T \left(\sum_{k=1}^s d_{ijk} F_k \right) \gamma \\ &= -\beta^T (F_i D_j + F_j D_i) \gamma \\ &= \beta^T F_i^T D_j \gamma - \gamma^T D_i F_j^T \beta \\ &= (\Theta^-(\beta)\Theta^+(\gamma) - \Theta^+(\gamma)\Theta^-(\beta))_{ij} \\ &= ([\Theta^-(\beta), \Theta^+(\gamma)])_{ij}. \end{aligned}$$

Finally, applying the same procedure but using (14) instead gives

$$\begin{aligned} (\Theta^+(\Theta^-(\beta)\gamma))_{ij} &= - \sum_{k=1}^s d_{ijk} \beta^T F_k \gamma \\ &= -\beta^T \left(\sum_{k=1}^s d_{ijk} F_k \right) \gamma \\ &= -\beta^T (D_i F_j + D_j F_i) \gamma \\ &= \beta^T D_i F_j^T \gamma - \gamma^T F_i^T D_j \beta \\ &= (\Theta^+(\beta)\Theta^-(\gamma) - \Theta^-(\gamma)\Theta^+(\beta))_{ij} \\ &= ([\Theta^+(\beta), \Theta^-(\gamma)])_{ij}, \end{aligned}$$

which completes the proof. \blacksquare

Lemma 2: Let $A, B \in \mathbb{C}^{n_w \times s}$, and denote by A_i, B_i their respective rows, $i = 1, \dots, n_w$. Then

$$[x, (Ax)^T] = -2\mathbf{i} (\Theta^-(A_1)x, \dots, \Theta^-(A_{n_w})x), \quad (20a)$$

$$\begin{aligned} [x, (Ax)^T] Bx &= -2\mathbf{i} \sum_{k=1}^{n_w} \left(\frac{2}{n} \Theta^-(A_k) B_k^T \right. \\ &\quad \left. + \Theta^-(A_k) \Theta^+(B_k)x - \mathbf{i} \Theta^-(A_k) \Theta^-(B_k)x \right), \quad (20b) \end{aligned}$$

$$(Bx)^T[Ax, x^T] = 2\mathbf{i} \sum_{k=1}^{n_w} \left(\frac{2}{n} \Theta^-(A_k) B_k^T + \Theta^-(A_k) \Theta^+(B_k) x + \mathbf{i} \Theta^-(A_k) \Theta^-(B_k) x \right)^T. \quad (20c)$$

Proof: The goal is to rewrite (20a) in terms of $[x, x^T]$ and $\{x, x^T\}$ in order to apply (17) and (18). Then

$$\begin{aligned} [x, (Ax)^T] &= \begin{pmatrix} x_1 x^T A_1^T & \cdots & x_1 x^T A_{n_w}^T \\ \vdots & \ddots & \vdots \\ x_s x^T A_1^T & \cdots & x_s x^T A_{n_w}^T \end{pmatrix} \\ &\quad - \begin{pmatrix} A_1 x x_1 & \cdots & A_1 x x_1 \\ \vdots & \ddots & \vdots \\ A_{n_w} x x_1 & \cdots & A_{n_w} x x_s \end{pmatrix}^T \\ &= (x x^T A_1^T - ((A_1 x) x^T)^T, \dots, x x^T A_{n_w}^T - ((A_{n_w} x) x^T)^T) \\ &= (x x^T A_1^T - (x x^T)^T A_1^T, \dots, x x^T A_{n_w}^T - (x x^T)^T A_{n_w}^T) \\ &= ([x, x^T] A_1^T, \dots, [x, x^T] A_{n_w}^T) \\ &= 2\mathbf{i} (\Theta^-(x) A_1^T, \dots, \Theta^-(x) A_{n_w}^T). \end{aligned}$$

Thus, Lemma 1 gives

$$[x, (Ax)^T] = -2\mathbf{i} (\Theta^-(A_1)x, \dots, \Theta^-(A_{n_w})x).$$

For (20b), note that the scalar operator $B_i x$ commutes with $\Theta^-(A_j)$ for any i and j . Recall that

$$x x^T = \frac{1}{2}([x, x^T] + \{x, x^T\}).$$

It then follows that

$$\begin{aligned} [x, (Ax)^T] Bx &= -2\mathbf{i} (\Theta^-(A_1)x, \dots, \Theta^-(A_{n_w})x) \begin{pmatrix} B_1 x \\ \vdots \\ B_{n_w} x \end{pmatrix} \\ &= -2\mathbf{i} \begin{pmatrix} A_1 F_1^T x & \cdots & A_{n_w} F_1^T x \\ \vdots & \ddots & \vdots \\ A_1 F_s^T x & \cdots & A_{n_w} F_s^T x \end{pmatrix} \begin{pmatrix} B_1 x \\ \vdots \\ B_{n_w} x \end{pmatrix} \\ &= -2\mathbf{i} \sum_{k=1}^{n_w} \begin{pmatrix} A_k F_1^T x x^T B_k^T \\ \vdots \\ A_k F_s^T x x^T B_k^T \end{pmatrix} \\ &= -2\mathbf{i} \sum_{k=1}^{n_w} \begin{pmatrix} A_k F_1^T \left(\frac{2}{n} I_s + \Theta^+(x) + \mathbf{i} \Theta^-(x) \right) B_k^T \\ \vdots \\ A_k F_s^T \left(\frac{2}{n} I_s + \Theta^+(x) + \mathbf{i} \Theta^-(x) \right) B_k^T \end{pmatrix} \\ &= -2\mathbf{i} \sum_{k=1}^{n_w} \left(\frac{2}{n} \Theta^-(A_k) B_k^T + \Theta^-(A_k) \Theta^+(x) B_k^T + \mathbf{i} \Theta^-(A_k) \Theta^-(x) B_k^T \right) \\ &= -2\mathbf{i} \sum_{k=1}^{n_w} \left(\frac{2}{n} \Theta^-(A_k) B_k^T + \Theta^-(A_k) \Theta^+(B_k) x - \mathbf{i} \Theta^-(A_k) \Theta^-(B_k) x \right). \end{aligned}$$

Similarly for (20c), one has, applying (1), that

$$\begin{aligned} (Bx)^T[Ax, x^T] &= 2\mathbf{i} \sum_{k=1}^{n_w} (B_k x A_k F_1^T x, \dots, B_k x A_k F_s^T x) \\ &= 2\mathbf{i} \sum_{k=1}^{n_w} (B_k x \Theta_1^-(A_k) x, \dots, B_k x \Theta_s^-(A_k) x) \\ &= 2\mathbf{i} \sum_{k=1}^{n_w} (\Theta_1^-(A_k) (x x^T)^T B_k^T, \dots, \Theta_s^-(A_k) (x x^T)^T B_k^T) \\ &= 2\mathbf{i} \sum_{k=1}^{n_w} \left(\Theta^-(A_k) \left(\frac{2}{n} I_s + \Theta^+(x) - \mathbf{i} \Theta^-(x) \right) B_k^T \right)^T \\ &= 2\mathbf{i} \sum_{k=1}^{n_w} \left(\frac{2}{n} \Theta^-(A_k) B_k^T + \Theta^-(A_k) \Theta^+(x) B_k^T - \mathbf{i} \Theta^-(A_k) \Theta^-(x) B_k^T \right)^T \\ &= 2\mathbf{i} \sum_{k=1}^{n_w} \left(\frac{2}{n} \Theta^-(A_k) B_k^T + \Theta^-(A_k) \Theta^+(B_k) x + \mathbf{i} \Theta^-(A_k) \Theta^-(B_k) x \right)^T. \end{aligned}$$

■

The explicit computation of the vector fields in (2) is given in the next lemma.

Lemma 3: The component coefficients of equations (2) and (3) are

$$[x, \mathcal{H}] = -2\mathbf{i} \Theta^-(\alpha) x, \quad (21a)$$

$$[x, L^T] = -2\mathbf{i} (\Theta^-(\Lambda_1)x, \dots, \Theta^-(\Lambda_{n_w})x), \quad (21b)$$

$$[x, L^\dagger] = -2\mathbf{i} (\Theta^-(\Lambda_1^\#)x, \dots, \Theta^-(\Lambda_{n_w}^\#)x), \quad (21c)$$

$$\begin{aligned} [L^\#, x^T] L &= \sum_{k=1}^{n_w} \left(\frac{4\mathbf{i}}{n} \Theta^-(\Lambda_k^\#) \Lambda_k^T + 2\mathbf{i} \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k) x + 2\Theta^-(\Lambda_k^\#) \Theta^-(\Lambda_k) x \right), \quad (21d) \end{aligned}$$

$$\begin{aligned} (L^\dagger [x, L^T]^T)^T &= \sum_{k=1}^{n_w} \left(\frac{4\mathbf{i}}{n} \Theta^-(\Lambda_k^\#) \Lambda_k^T - 2\mathbf{i} \Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) x + 2\Theta^-(\Lambda_k) \Theta^-(\Lambda_k^\#) x \right). \quad (21e) \end{aligned}$$

Proof: Commutators (21a)-(21c) follow directly from (20a). Commutator (21d) is computed out of (20b) as

$$\begin{aligned} [L^\#, x^T] L &= -[x, L^\dagger] L \\ &= \sum_{k=1}^{n_w} \left(\frac{4\mathbf{i}}{n} \Theta^-(\Lambda_k^\#) \Lambda_k^T + 2\mathbf{i} \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k) x + 2\Theta^-(\Lambda_k^\#) \Theta^-(\Lambda_k) x \right). \end{aligned}$$

Finally, commutator (21e) is obtained using (20c) as

$$(L^\dagger [x, L^T]^T)^T = - (L^\dagger [L^\#, x^T])^T$$

$$= \sum_{k=1}^{n_w} \left(\frac{4\mathbf{i}}{n} \Theta^-(\Lambda_k^\#) \Lambda_k^T \right. \\ \left. - 2\mathbf{i} \Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) x + 2\Theta^-(\Lambda_k) \Theta^-(\Lambda_k^\#) x \right)$$

IV. PHYSICAL REALIZABILITY

The main contribution of the paper are given in this section. First, physical realizability is introduced next

Definition 3: A system described by equations (9) and (10) is said to be physically realizable if there exist \mathcal{H} and L such that (9) can be written as in (2).

The explicit form of matrices $A_0, A, B_{1k}, B_{2k}, C_1$ and C_2 in terms of the Hamiltonian and coupling operator is given next.

Theorem 1: Let $\mathcal{H} = \alpha x$, with $\alpha^T \in \mathbb{R}^s$, and $L = \Lambda x$, with $\Lambda \in \mathbb{C}^{n_w \times s}$. Then

$$A_0 = \frac{4\mathbf{i}}{n} \sum_{k=1}^{n_w} \Theta^-(\Lambda_k^\#) \Lambda_k^T, \quad (22a)$$

$$A = -2\Theta^-(\alpha) + \sum_{k=1}^{n_w} (R_k - \mathbf{i}Q_k), \quad (22b)$$

$$B_{1k} = \Theta^-(\mathbf{i}(\Lambda_k^\# - \Lambda_k)), \quad (22c)$$

$$B_{2k} = \Theta^-(\Lambda_k + \Lambda_k^\#), \quad (22d)$$

$$C_1 = \Lambda + \Lambda^\#, \quad (22e)$$

$$C_2 = \mathbf{i}(\Lambda^\# - \Lambda), \quad (22f)$$

where

$$R_k \triangleq \Theta^-(\Lambda_k) \Theta^-(\Lambda_k^\#) + \Theta^-(\Lambda_k^\#) \Theta^-(\Lambda_k)$$

and

$$Q_k \triangleq \Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k).$$

Proof: The proof follows by direct application of Lemmas 1, 2 and 3. That is, equation (2) is re-written using (21a)-(21e) as the following bilinear QSDE

$$dx = -2\Theta^-(\alpha)x dt + \frac{4\mathbf{i}}{n} \sum_{k=1}^{n_w} \Theta^-(\Lambda_k^\#) \Lambda_k^T dt \\ + \sum_{k=1}^{n_w} \left(\Theta^-(\Lambda_k) \Theta^-(\Lambda_k^\#) + \Theta^-(\Lambda_k^\#) \Theta^-(\Lambda_k) \right) x dt \\ - \mathbf{i} \sum_{k=1}^{n_w} \left(\Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k) \right) x dt \\ + \mathbf{i} \left(\Theta^-(\Lambda_1^\# - \Lambda_1)x, \dots, \Theta^-(\Lambda_{n_w}^\# - \Lambda_{n_w})x \right) d\bar{W}_1 \\ + \left(\Theta^-(\Lambda_1 + \Lambda_1^\#)x, \dots, \Theta^-(\Lambda_{n_w} + \Lambda_{n_w}^\#)x \right) dW. \quad (23)$$

Also, as mentioned in Section II, the output fields \bar{Y}_1 and \bar{Y}_2 depend linearly on L, L^\dagger and the input fields \bar{W}_1 and \bar{W}_2 , i.e.,

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} \Lambda + \Lambda^\# \\ \mathbf{i}(\Lambda^\# - \Lambda) \end{pmatrix} x dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}.$$

It is now easy to identify matrices $A_0, A, B_{1k}, B_{2k}, C_1$ and C_2 , which ends the proof. \blacksquare

Note that all matrices involved in the above equation are real. To confirm that, observe that $\Lambda^\# - \Lambda$ is purely imaginary and $\Lambda + \Lambda^\#$ is purely real. Now fix k and compute the real part of $\Theta^-(\Lambda_k) \Lambda_k^\dagger$ and $\Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k)$. Given that $(\Theta^-(\Lambda_k) \Lambda_k^\dagger)^\# = -\Theta^-(\Lambda_k) \Lambda_k^\dagger$, one has that

$$\text{Re}\{\Theta^-(\Lambda) \Lambda^\dagger\} = \frac{1}{2} (\Theta^-(\Lambda) \Lambda^\dagger + (\Theta^-(\Lambda) \Lambda^\dagger)^\#) = 0.$$

Also,

$$\begin{aligned} & \text{Re}\{\Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k)\} \\ &= \frac{1}{2} \left(\Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k) \right. \\ & \quad \left. + \left(\Theta^-(\Lambda_k) \Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#) \Theta^+(\Lambda_k) \right)^\# \right) = 0. \end{aligned}$$

Moreover, by direct inspection $B_{ik}^T = -B_{ik}$ for $i = 1, 2$ and $k = 1, \dots, n_w$.

Now, from a control perspective, it is necessary to characterize when a bilinear QSDE posses underlying Hamiltonian and coupling operators which allows to express the matrices comprising (9) and (10) as in Theorem 1. Thus, the second and most relevant result of the paper is given in the next theorem, which establishes necessary and sufficient conditions for the physical realizability of a bilinear QSDE.

Theorem 2: System (9) with output equation (10) is physically realizable if and only if

$$i. A_0 = \frac{1}{n} \sum_{k=1}^{n_w} (\mathbf{i}B_{1k} + B_{2k}) ((C_1)_k + \mathbf{i}(C_2)_k)^T,$$

$$ii. B_{1k} = \Theta^-((C_2)_k),$$

$$iii. B_{2k} = \Theta^-((C_1)_k),$$

$$iv. A + A^T + \sum_{i,k=1}^{2,n_w} B_{ik} B_{ik}^T = \frac{n}{2} \Theta^+(A_0),$$

where $(C_i)_k$ indicates the k -th row of C_i . In which case, the coupling matrix can be identified to be

$$\Lambda = \frac{1}{2}(C_1 + \mathbf{i}C_2),$$

and α , defining the system Hamiltonian, is

$$\alpha = \frac{1}{4n} \text{vec} \left(A^T - A + \frac{1}{2} \sum_{k=1}^{n_w} ([B_{2k}, \Theta^+((C_2)_k)] \right. \\ \left. - [B_{1k}, \Theta^+((C_1)_k)]) \right)^T F. \quad (24)$$

Proof: Assuming that (9) and (10) are physically realizable implies that (22a)-(22f) are satisfied. By comparison, conditions (ii)-(iii) hold. Condition (i) is written from (22e) and (22f) as

$$\begin{aligned} A_0 &= \frac{\mathbf{i}}{n} \sum_{k=1}^{n_w} \Theta^-((C_1)_k - \mathbf{i}(C_2)_k) ((C_1)_k + \mathbf{i}(C_2)_k)^T \\ &= \frac{1}{n} \sum_{k=1}^{n_w} (\mathbf{i}B_{1k} + B_{2k}) ((C_1)_k + \mathbf{i}(C_2)_k)^T \end{aligned}$$

Now, one has that

$$\begin{aligned} B_{1k}B_{1k}^T &= \Theta^-(\Lambda_k^\# - \Lambda_k)^2 \\ &= \Theta^-(\Lambda_k^\#)\Theta^-(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#)\Theta^-(\Lambda_k) \\ &\quad - \Theta^-(\Lambda_k)\Theta^-(\Lambda_k^\#) + \Theta^-(\Lambda_k)\Theta^-(\Lambda_k). \end{aligned}$$

Similarly,

$$\begin{aligned} B_{2k}B_{2k}^T &= -\Theta^-(\Lambda_k + \Lambda_k^\#)^2 \\ &= -\Theta^-(\Lambda_k^\#)\Theta^-(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#)\Theta^-(\Lambda_k) \\ &\quad - \Theta^-(\Lambda_k)\Theta^-(\Lambda_k^\#) - \Theta^-(\Lambda_k)\Theta^-(\Lambda_k). \end{aligned}$$

Thus, $B_{1k}B_{1k}^T + B_{2k}B_{2k}^T = -2R_k$. One can now rewrite A in terms of α , B_{1k} and B_{2k} as

$$\begin{aligned} A &= -2\Theta^-(\alpha) \\ &\quad - \frac{1}{2} \sum_{k=1}^{n_w} (B_{1k}B_{1k}^T + B_{2k}B_{2k}^T) - \mathbf{i}Q_k. \end{aligned} \quad (25)$$

Similarly,

$$\begin{aligned} A^T &= 2\Theta^-(\alpha) \\ &\quad - \frac{1}{2} \sum_{k=1}^{n_w} (B_{1k}B_{1k}^T + B_{2k}B_{2k}^T) - \mathbf{i}Q_k^T. \end{aligned} \quad (26)$$

Adding (25) and (26) gives

$$A + A^T = - \sum_{k=1}^{n_w} (B_{1k}B_{1k}^T + B_{2k}B_{2k}^T) - \mathbf{i}(Q_k + Q_k^T).$$

The (i, j) component of $Q_k + Q_k^T$ is computed as

$$\begin{aligned} (Q_k + Q_k^T)_{ij} &= \Theta^-(\Lambda_k)\Theta^+(\Lambda_k^\#) - \Theta^-(\Lambda_k^\#)\Theta^+(\Lambda_k) \\ &\quad - \Theta^+(\Lambda_k^\#)\Theta^-(\Lambda_k) + \Theta^+(\Lambda_k)\Theta^-(\Lambda_k^\#) \\ &= -\Lambda_k^T F_i D_j \Lambda_k^\# + \Lambda_k^\dagger F_i D_j \Lambda_k \\ &\quad - \Lambda_k^\dagger D_i F_j \Lambda_k + \Lambda_k^T D_i F_j \Lambda_k^\#. \end{aligned}$$

Note that every summand is a scalar, which is equal to its transpose. By (13) and (14), it follows that

$$\begin{aligned} (Q_k + Q_k^T)_{ij} &= \Lambda_k^T (D_i F_j + D_j F_i) \Lambda_k^\# + \Lambda_k^T (F_i D_j + F_j D_i) \Lambda_k^\# \\ &= \Lambda_k^T \sum_{k=1}^s d_{ijk} F_k \Lambda_k^\# + \Lambda_k^T \sum_{k=1}^s d_{ijk} F_k \Lambda_k^\# \\ &= 2 \sum_{k=1}^s d_{ijk} \Lambda_k^T D_k \Lambda_k^\# \\ &= 2 \left(\Theta^+(\Theta^-(\Lambda_k) \Lambda_k^\#) \right)_{ij} \\ &= -2 \left(\Theta^+(\Theta^-(\Lambda_k^\#) \Lambda_k) \right)_{ij} \\ &= \frac{n\mathbf{i}}{2} \left(\Theta^+(A_0) \right)_{ij}. \end{aligned}$$

Therefore, adding (25) and (26) gives

$$A + A^T + \sum_{i,k=1}^{2,n_w} B_{ik}B_{ik}^T = \frac{n}{2} \Theta^+(A_0),$$

which is condition (iv). Conversely, one needs to show that if conditions (i)-(iv) of Theorem 2 are satisfied, then there exist matrices α and Λ such that system (9) is physically

realizable. Let

$$\begin{aligned} \Theta^-(\alpha) &\triangleq \frac{1}{4} \left(A^T - A \right. \\ &\quad \left. + \frac{1}{2} \sum_{k=1}^{n_w} ([B_{2k}, \Theta^+((C_2)_k)] - [B_{1k}, \Theta^+((C_1)_k)]) \right). \end{aligned} \quad (27)$$

It is trivial to check that the right-hand-side of (27) is antisymmetric and hence this equation uniquely defines α via (15). Also, let

$$\Lambda = \frac{1}{2}(C_1 + \mathbf{i}C_2). \quad (28)$$

Then, Q_k can be written in terms of B_1 , B_2 , C_1 , and C_2 as follows

$$\begin{aligned} Q_k &= \frac{1}{4} \left(\Theta^-(C_1 + \mathbf{i}C_2)_k \Theta^+(C_1 - \mathbf{i}C_2)_k \right. \\ &\quad - \Theta^-(C_1 - \mathbf{i}C_2)_k \Theta^+(C_1 + \mathbf{i}C_2)_k \\ &\quad - \Theta^+(C_1 + \mathbf{i}C_2)_k \Theta^-(C_1 - \mathbf{i}C_2)_k \\ &\quad \left. + \Theta^+(C_1 - \mathbf{i}C_2)_k \Theta^-(C_1 + \mathbf{i}C_2)_k \right) \\ &= -\frac{\mathbf{i}}{2} \left(\Theta^-(C_1)_k \Theta^+(C_2)_k \right. \\ &\quad \left. - \Theta^-(C_2)_k \Theta^+(C_1)_k \right). \end{aligned}$$

From (ii) and (iii), it follows that

$$\begin{aligned} Q_k &= \Theta^-(\Lambda_k)\Theta^-(\Lambda_k^\#) + \Theta^-(\Lambda_k^\#)\Theta^-(\Lambda_k) \\ &= -\frac{\mathbf{i}}{2} (B_{2k}\Theta^+((C_2)_k) - B_{1k}\Theta^+((C_1)_k)). \end{aligned}$$

Then,

$$Q_k^T - Q_k = \frac{\mathbf{i}}{2} ([B_{2k}, \Theta^+((C_2)_k)] - [B_{1k}, \Theta^+((C_1)_k)]).$$

Similarly, it is simple to write R_k in terms of C_1 and C_2 . That is,

$$\begin{aligned} R_k &= \frac{1}{4} \left(\Theta^-(C_1 + \mathbf{i}C_2)_k \Theta^-(C_1 - \mathbf{i}C_2)_k \right. \\ &\quad \left. + \Theta^-(C_1 - \mathbf{i}C_2)_k \Theta^-(C_1 + \mathbf{i}C_2)_k \right) \\ &= \frac{1}{4} \left(\Theta^-(C_1)_k \Theta^-(C_1)_k - \mathbf{i} \Theta^-(C_1)_k \Theta^-(C_2)_k \right. \\ &\quad \left. + \mathbf{i} \Theta^-(C_2)_k \Theta^-(C_1)_k + \Theta^-(C_2)_k \Theta^-(C_2)_k \right) \\ &\quad + \Theta^-(C_1)_k \Theta^-(C_1)_k + \mathbf{i} \Theta^-(C_1)_k \Theta^-(C_2)_k \\ &\quad - \mathbf{i} \Theta^-(C_2)_k \Theta^-(C_1)_k + \Theta^-(C_2)_k \Theta^-(C_2)_k \\ &= \frac{1}{2} \left(\Theta^-(C_1)_k \Theta^-(C_1)_k + \Theta^-(C_2)_k \Theta^-(C_2)_k \right). \end{aligned}$$

It clear that R_k is symmetric. From (ii) and (iii), one obtains that

$$\begin{aligned} R_k &= \Theta^-(\Lambda_k)\Theta^-(\Lambda_k^\#) + \Theta^-(\Lambda_k^\#)\Theta^-(\Lambda_k) \\ &= -\frac{1}{2} (B_{1k}B_{1k}^T + B_{2k}B_{2k}^T). \end{aligned}$$

From (i) and (28),

$$\Theta^+(A_0) = -\frac{2\mathbf{i}}{n} (Q_k + Q_k^T)$$

Since (iv) implies

$$A^T = -A - \sum_{i,k=1}^{2,n_w} B_{ik}B_{ik}^T + \frac{n}{2} \Theta^+(A_0),$$

one can use (27) to obtain

$$\begin{aligned}
& \Theta^-(\alpha) \\
&= \frac{1}{4} \left(-2A + \frac{n}{2} \Theta^+(A_0) - \sum_{k=1}^{n_w} \left(B_{1k} B_{1k}^T + B_{2k} B_{2k}^T \right. \right. \\
&\quad \left. \left. + \frac{1}{2} ([B_{2k}, \Theta^+((C_2)_k)] - [B_{1k}, \Theta^+((C_1)_k)]) \right) \right) \\
&= -\frac{1}{2} A - \frac{1}{4} \sum_{k=1}^{n_w} (2R_k - \mathbf{i}(Q_k - Q_k^T) - \mathbf{i}(Q_k^T - Q_k)) \\
&= -\frac{1}{2} A - \frac{1}{2} \sum_{k=1}^{n_w} (R_k - \mathbf{i}Q_k),
\end{aligned}$$

which is equivalent to (22b). Moreover, using (19), (27) and applying the stacking operator to $\Theta^-(\alpha)$, α is explicitly obtained as

$$(F^T \text{vec}(\Theta^-(\alpha)))^T = (F^T F \alpha^T)^T = n\alpha.$$

Hence,

$$\begin{aligned}
\alpha = \frac{1}{4n} \text{vec} \left(A^T - A + \frac{1}{2} \sum_{k=1}^{n_w} [B_{2k}, \Theta^+((C_2)_k)] \right. \\
\left. - [B_{1k}, \Theta^+((C_1)_k)] \right)^T F,
\end{aligned}$$

which completes the proof. \blacksquare

Note that the physical realizability conditions do not require the computation of the Hamiltonian (24), which depends on the structure constants d and f , in order to know whether or not the system given by equations (9) and (10) is quantum.

V. CONCLUSIONS

A condition for physical realizability was given for open multi-level quantum systems. Under this condition it was

shown that there exist operators \mathcal{H} and L such that the bilinear QSDE (9) with output equation (10) can be written as in (2). This condition used explicitly the algebra generated by $SU(n)$. Moreover, the interaction of the system with multiple quantum fields was introduced to the formalism.

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